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Short Communication

A preconditioned iterative method for modal frequency-response analysis of structures with non-proportional damping

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Abstract

A new stationary-type iterative method, the hybrid Jacobi iterative method, is developed for solving modal frequency-response problems with non-proportional damping, which is indefinite linear system. The hybrid Jacobi iterative method is derived by introducing a new preconditioning matrix that results in combining the Jacobi iterative method with the block Jacobi iterative method for solving an indefinite linear system. Numerical demonstrations show accurate results and high performance compared to the direct method.

1. Introduction

The frequency-response analysis (FRA) is one of the most popular methods in the industry for large and complex structural vibration analysis. Because the FRA in terms of all of the finite element (FE) degrees of freedom is expensive, industry has mainly used the modal FRA. Some challenging aspects of performing the modal FRA arise from non-proportional damping [1,2].

The standard approach for solving the coupled modal frequency-response problem that includes the non-proportional damping [3], which is not proportional to the mass and/or stiffness matrices, is to use either direct methods or iterative methods [3]. Direct methods are the most straightforward and accurate, but are expensive due to factorization costs that are $O(m^3)$ operations [4]. An iterative method is a natural alternative to a direct method for solving a large-scale linear system with a fully populated coefficient matrix. Iterative methods have more speed advantages than direct methods, but the convergence rate of iterative methods depends on spectral properties of the coefficient matrix. Therefore, a preconditioner is used to transform the original matrix to an equivalent matrix which has more favorable spectral properties. A good preconditioner makes the original system well conditioned and easy to solve. However, applying a preconditioner requires some extra effort both for initial setup and for applying it in each iteration [4–6].

There are two categories of iterative methods [5]. Stationary methods are older, and simpler to understand and implement, but usually not as effective. The following methods are in this category: Jacobi, Gauss-Seidel, SOR, and SSOR method. In particular, these stationary iterative methods are effective for positive definite

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linear system, but may break down for indefinite linear systems [6]. Non-stationary methods are a relatively recent development based on a sequence of orthogonal vectors. The BiConjugate Gradient (BiCG) [5], the Generalized Minimal Residual (GMRES) [7], and the Quasi-Minimal Residual (QMR) [8] methods are some of the typical algorithms in this category. Their analysis is effective, but usually harder to understand.

This paper presents a stationary-type iterative method for solving the modal frequency-response problem with non-proportional damping, which results in a complex indefinite linear system. In stationary methods, the Jacobi method has the simplest preconditioner that consists of just the diagonal elements of the matrix. However, a breakdown of the algorithm may occur corresponding to a zero pivot during the factorization of a tridiagonal matrix if the matrix is indefinite. Instead, the Block Jacobi method has a block-diagonal matrix. The Block Jacobi method has more advantages over the Jacobi method [6]. However, division operations are usually quite costly since it needs factorization of each block for inversion.

In this paper, the hybrid Jacobi method is introduced with a new preconditioner for the damped modal frequency-response problem. The numerical demonstrations are performed to investigate the accuracy and convergence rate of the hybrid Jacobi method.

2. Modal frequency-response problem formulation

The frequency-response problem in the FE space can be written in the form [9]

$$[-\omega^2 M + \mathrm{i}\omega B + (1+\mathrm{i}\gamma)K + \mathrm{i}K_s]x(\omega) = p(\omega), \tag{1}$$

where M, B, K and $K_s \in \mathbb{R}^{n\times n}$ are the FE mass, viscous damping, stiffness and local structural damping matrices, respectively. The scalar γ is the global structural damping level and ω is the excitation frequency. $p \in \mathbb{C}^n$ is the load vector and $x \in \mathbb{C}^n$ is the displacement vector.

With the partial eigensolution of the generalized eigenvalue problem $K\Phi = M\Phi\Lambda$, the modal frequency-response problem is obtained in the form [9]

$$[-\omega^2 I + \mathrm{i}\omega\bar{B} + (1+\mathrm{i}\gamma)\Lambda + \mathrm{i}\bar{K}_s]z(\omega) = f(\omega) \tag{2}$$

in which the size of matrices are m ($m \ll n$), which is the number of modes obtained from $K\Phi = M\Phi\Lambda$. Using orthogonality relationships and mass normalization [3], the mass and stiffness matrices are diagonalized. Note that $\bar{B} = \Phi^{\mathrm{T}}B\Phi$, $\bar{K}_s = \Phi^{\mathrm{T}}K_s\Phi \in \mathbb{R}^{m\times m}$ are generally fully populated and $f = \Phi^{\mathrm{T}}p \in \mathbb{C}^m$. Once the modal solution $z(\omega)$ is obtained over the desired frequency range, the FE solutions can be obtained from the backtransformation $x(\omega) = \Phi z(\omega)$.

3. An iterative methods for damped modal frequency-response analysis

3.1. Hybrid Jacobi method

For simplicity, Eq. (2) is denoted as

$$A(\omega)z = f \tag{3}$$

in which $A(\omega) = -\omega^2 I + \mathrm{i}\omega\bar{B} + (1+\mathrm{i}\gamma)\Lambda + \mathrm{i}\bar{K}_s$. Note that the coefficient matrix A is a complex indefinite matrix and is frequency dependent. The diagonal terms $(-\omega^2 I + (1+\mathrm{i}\gamma)\Lambda)$ are a dominant part in the matrix. Usually, the value of γ is much less than one and the eigenvalue of Λ is much greater than one, so that the real part of the diagonal terms is the most dominant in the coefficient matrix.

The hybrid Jacobi method is derived by combining the Jacobi method [5] with the block Jacobi method [6] to solve the indefinite linear system. In a stationary iterative method, the original problem is rewritten in the equivalent form

$$Mz^{k+1} = Nz^k + f \tag{4}$$

in which $A = M - N$ and the superscript k is the iteration count. The number of iterations depends on the spectral properties of the coefficient matrix, so that preconditioner is essential. With a good preconditioner M,

$M^{-1}A$ becomes well conditioned. In the Jacobi method, the preconditioner M consists of just the diagonal elements of the matrix. For the block Jacobi method, it has a block-diagonal matrix in M.

In the Hybrid Jacobi method, the new preconditioner or splitting matrix M is defined by noting the coefficient matrix characteristic. First, A, z and f are partitioned in the following way:

$$A(\omega)z = \left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right] \left\{\begin{array}{c} z_1 \\ \hline z_2 \\ \hline z_3 \end{array}\right\} = \left\{\begin{array}{c} f_1 \\ \hline f_2 \\ \hline f_3 \end{array}\right\}. \tag{5}$$

This splitting is made by noting the characteristic of the real part of the diagonal term $(-\omega^2 I + (1 + i\gamma)\Lambda)$ that is the dominant term in the coefficient matrix. $A_{11} \in \mathbb{C}^{m_1 \times m_1}$ is where eigenvalues in Λ are clearly less than ω^2, and for $A_{33} \in \mathbb{C}^{m_3 \times m_3}$ they are clearly greater than ω^2. This means that A_{11} and A_{33} become the negative definite and positive definite parts, respectively. Assume that the eigenvalues in Λ are in ascending order. The remaining part $A_{22} \in \mathbb{C}^{m_2 \times m_2}$ has diagonal elements close to zero because eigenvalues in Λ are close to ω^2. Off-diagonal terms are relatively small compared to diagonal elements. The partition of the matrix $A \in \mathbb{C}^{m \times m}$ depends on the excitation frequency ω, and $m = m_1 + m_2 + m_3$. Note that the size of A_{22} is much smaller than that of A_{11} and A_{33}, that is, $m_2 \ll m_1, m_3$.

Based on this partitioning, the preconditioner M is defined as

$$M = \begin{bmatrix} \mathrm{diag}(A_{11}) & & 0 \\ & A_{22} & \\ 0 & & \mathrm{diag}(A_{33}) \end{bmatrix}. \tag{6}$$

This preconditioner makes use of the Jacobi method for A_{11}, A_{33}, which is a negative and a positive definite part of the matrix, and the block Jacobi method for A_{22} that is an indefinite part of the matrix. This preconditioner takes little storage and easy to implement since the size of A_{22} is small compared to the size of A_{11} and A_{33}.

The brief algorithm of the Hybrid Jacobi method is shown in Fig. 1. Step (1) computes the initial solution. Eqs. (1.1) and (1.3) are economical computations since A_{11} and A_{33} are diagonal matrices. To solve (1.2), the symmetric indefinite linear solver algorithm, the Bunch–Kaufman algorithm [11,12], is used. Since A_{22} is a relatively small matrix, $m_2 \ll m_1, m_3$, we can solve the indefinite linear system inexpensively. Step (3)–(7) is the iteration process. For each ω, once A_{22} is factored in Step (1.2), it can be used again in the next iteration, which also saves the computational cost. This algorithm is called at each excitation frequency ω to solve the corresponding damped modal FRA problem.

For each excitation frequency ω

(1) initial $\eta^0 = \{\eta_1^0, \eta_1^0, \eta_1^0\}^T$ by solving $M\eta^0 = f$

(1.1) $\eta_1^0 = f/\mathrm{diag}(A_{11})$

(1.2) $\eta_2^0 = \mathrm{inv}(A_{22})f$

(1.3) $\eta_3^0 = f/\mathrm{diag}(A_{33})$

(2) calculate residual $r = -N\eta^0$

do while

(3) solve $Md\eta = r$, in which $d\eta = \{d\eta_1^k, d\eta_2^k, d\eta_3^k\}^T$

(3.1) $d\eta_1^k = r_1/\mathrm{diag}(A_{11})$

(3.2) $d\eta_2^k = \mathrm{inv}(A_{22})r_2$

(3.3) $d\eta_3^k = r_3/\mathrm{diag}(A_{33})$

(4) $r = -Nd\eta$

(5) check the convergence, exit

(6) $\eta = \eta + d\eta$

(7) iteration count $k = k + 1$

end

Fig. 1. Algorithm of the hybrid Jacobi method.

3.2. Partitioning for the preconditioner M

To ensure the convergence of the Hybrid Jacobi method, the spectral radius ρ, which is the maximum eigenvalue of $M^{-1}N$, should satisfy the following condition [4]:

$$\rho = \max\{\|\lambda\| : \lambda \in \lambda(M^{-1}N)\} < 1, \tag{7}$$

where λ represents the eigenvalue of $M^{-1}N$. To this end, the optimal partitioning of A should be selected [10].

To partition the matrix A, the position and dimension of the submatrix A_{22} are selected in the following way. First, the reference position ξ of the ξth eigenvalue λ_ξ in the eigenvalue matrix $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m)$, in which $\lambda_1 < \lambda_2 < \cdots < \lambda_m$, is defined as the position where the value of diagonal matrix $D(\omega) = (-\omega^2 I + \Lambda)$ is minimum. Then, with the reference position ξ, the lower and upper position $\alpha, \beta(\alpha < \beta)$ of the submatrix A_{22} are defined such that $|(\lambda_\xi/\lambda_\alpha) - r|$ is minimum and $|(\Lambda_\beta/\Lambda_\xi) - r|$ is maximum, respectively, for a given constant r ($r > 1$). The dimension of the submatrix A_{22} is $\beta - \alpha + 1$.

The position α and β vary at each excitation frequency ω because the position ξ also is different for the different ω. The constant r, which should be greater than one, is provided by the user. As the value r increases, the size of A_{22} also increases, which results in increasing convergence rate. However, the increasing performance is compensated by the increasing factorization cost of A_{22} in the step (1.2) of the algorithm described in Fig. 1. This is a typical trade-off between the convergence rate and the cost to set up the preconditioner in iterative method implementations [5].

4. Numerical example

An automobile panel FE model consists of 24,638 elements. The total degrees of freedom (n) is 147,828. In total, 830 modes (m) including 6 rigid body modes are obtained up to 1200 Hz. FRA analysis is performed up to 1000 Hz excitation frequency. IBM RS/6000 SP 200 MHz is used for the performance and accuracy measurement. The FE matrices are obtained from the commercial FE software NASTRAN [9].

Table 1 shows the elapsed time for the modal FRA with a different size of A_{22} matrix. For the comparison of accuracy and performance, a direct method, ZSYSV in LAPACK [12] that uses Bauch–Kaufman algorithm [4], is employed. It takes 33 min 51 s with the direct method. In the case of $r = 1.0$, which represents the conventional Jacobi iterative method that has only a diagonal matrix in M, the iterative method failed to converge. However, by introducing the preconditioner M defined in Eq. (6), the solution converges. The elapsed time of the iterative method is 5.8, 9.4, 9.9, and 11.0 times shorter than the direct method for the different $r = 1.05, 1.2, 1.4$, and 1.6.

Fig. 2 shows the dimension, and lower and upper position of the splitting matrix, A_{22}, for all excitation frequencies with different r. It shows that the size of A_{22} increases as the excitation frequency increases. Also, as the size of splitting matrix A_{22} or r increases, the convergence rate increases dramatically as illustrated in Fig. 3. Fig. 3 shows the relative norm of the residual of the hybrid Jacobi method for several different r at the excitation frequency 500 Hz. The relative norm of the residual is defined as $\|Az^k - f\|_2/\|f\|_2$. With $r = 1.2$ and 1.6, the solution is converged within 10 iterations. However, even after 100 iterations, the classical Jacobi method ($r = 1.0$) has not reached convergence.

Table 1
The elapsed time of the modal frequency response analysis with the hybrid Jacobi method with different r (unit [mm:ss])

r	Hybrid Jacobi
1.0	Fail
1.05	5:51
1.2	3:35
1.4	3:26
1.6	3:04

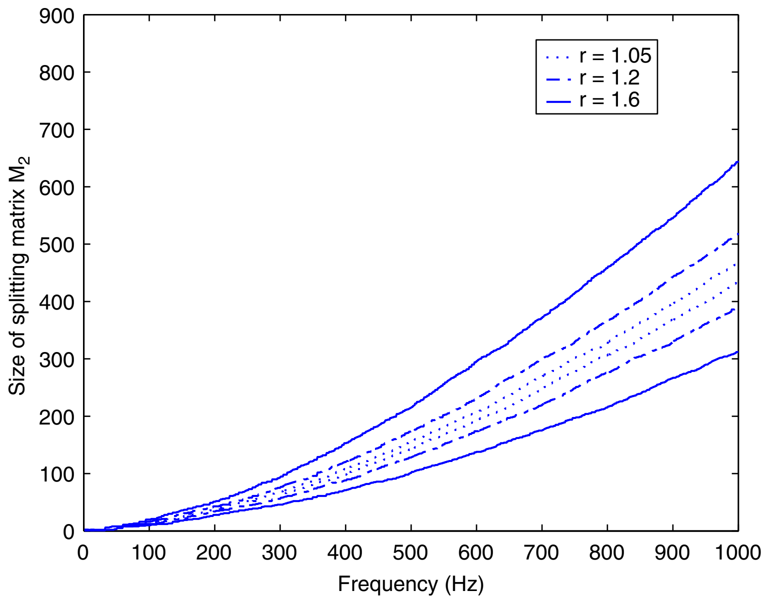


Fig. 2. The dimension, and lower and upper positions of the splitting matrix A_{22}.

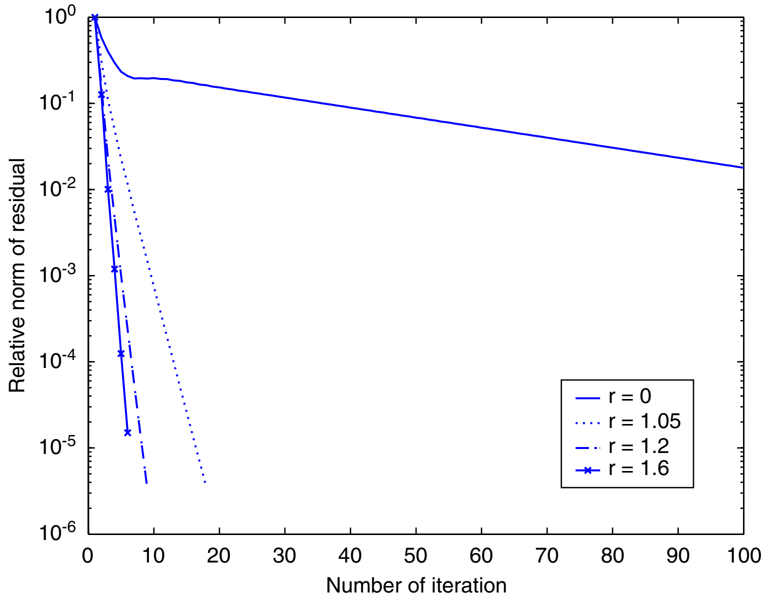
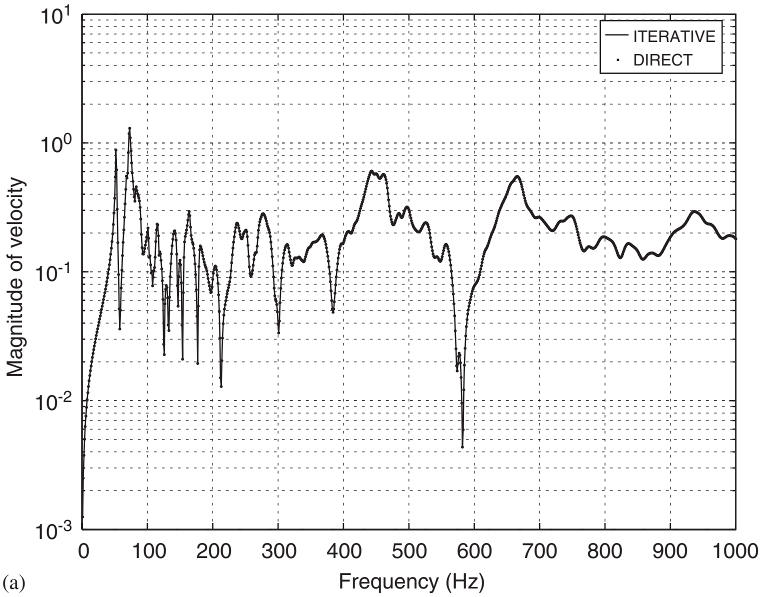


Fig. 3. Relative norm of residual of hybrid Jacobi method with different r.

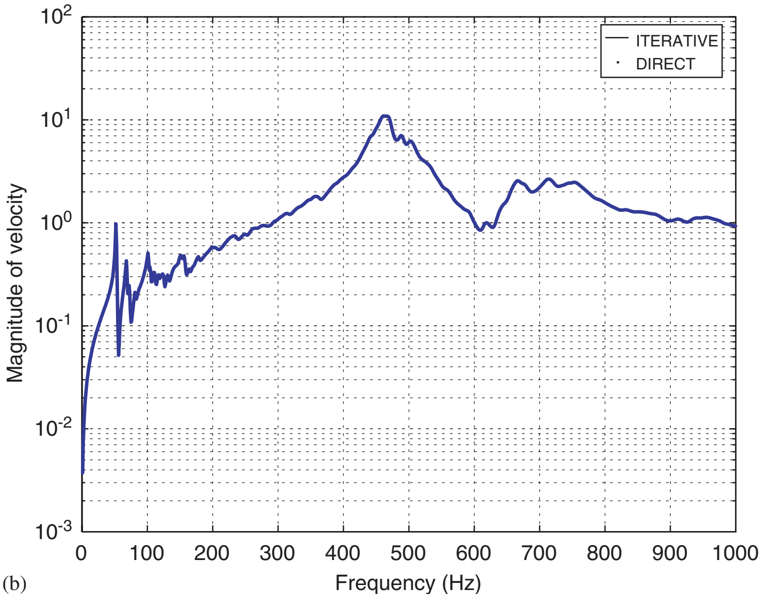
The frequency response from the hybrid Jacobi method with $r = 1.2$ and the direct method is plotted in Fig. 4(a) and (b), in which both frequency responses are indistinguishable. It shows that the iterative solution is accurate compared to the direct method solution.

5. Conclusion

In this study, a new stationary-type iterative method, the hybrid Jacobi method, is developed for solving modal frequency-response problems with non-proportional damping. A new preconditioner is introduced by



(a)



(b)

Fig. 4. Comparison of frequency response with the hybrid Jacobi method ($r = 1.2$) and direct method.

splitting the coefficient matrix into the negative definite and positive definite parts, and the indefinite part. This preconditioner takes little storage and is easy to implement since the indefinite part is small. The new preconditioner provides a significant convergence rate by combining the Jacobi iterative method for the negative definite, and positive definite parts, with the block Jacobi iterative method for the indefinite part. Numerical demonstrations show accurate result and high performance compared to the direct method.

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